

# SYMMETRICAL R-DUALS OF TYPE *III* AND A REPRESENTATION OF ITS FRAME OPERATORS

JAHANGIR CHESHMAVAR AND ALI AKBARNIA

**ABSTRACT.** In 2015, Stoeva and Christensen introduced different types of R-duals. A very important distinction of these types of R-duals is that type *III* is not symmetrical. However by choosing an appropriate sequence and applying mild technical change, we characterize symmetrical type *III* of R-duals. Given the associated frame of this symmetrical R-duals, we achieved a representation for square root of its inverse.

## 1. INTRODUCTION AND PRELIMINARIES

The Gabor systems in  $L^2(\mathbb{R})$  generated by parameters  $a, b > 0$  and a function  $g \in L^2(\mathbb{R})$  is the collection of functions

$$\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} := \{e^{2\pi imbx}g(x - na)\}_{m,n \in \mathbb{Z}}.$$

We refer the interested reader to the [7] for detailed treatments of Gabor systems. One of the important results in Gabor analysis is the duality principle, [6] states that a Gabor system is a frame if and only if the corresponding adjoint Gabor system is a Riesz sequence. With the purpose of generalizing and characterizing the duality principle to arbitrary separable Hilbert space  $\mathcal{H}$ , the concept of Riesz-duals (R-duals) of a certain sequence in  $\mathcal{H}$  were introduced in [1] by Casazza, Kutyniok and Lammers. For each sequence in  $\mathcal{H}$ , they construct a corresponding sequence dependent only on two orthonormal bases, with a kind of duality relation between them and present a general approach to derive duality principles in frame theory. In [3], the authors derived conditions for a sequence  $\{\omega_j\}_{j \in I}$  to be an R-dual of a given frame  $\{f_i\}_{i \in I}$ . In [4] Christensen et al. introduced the idea of R-duality in Banach spaces. Also in [5], the authors give an equivalent conditions of two sequences to be R-duals.

Let  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$  be orthonormal bases for separable Hilbert space  $\mathcal{H}$ . Let  $\{f_i\}_{i \in I} \subset \mathcal{H}$  be such that  $\sum_{i \in I} |\langle f_i, e_j \rangle|^2 < \infty, \forall j \in I$ .

---

*Key words and phrases.* Frames, Riesz sequence, Riesz basis, R-dual of type *I*, R-dual of type *III*.

*2010 Mathematics Subject Classification.* Primary 47A67; Secondary 42C15.

In [1] the R-dual of  $\{f_i\}_{i \in I}$  with respect to the orthonormal bases  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$  is defined as the sequence given by

$$\omega_j = \sum_{i \in I} \langle f_i, e_j \rangle h_i, \quad j \in I.$$

This R-dual is called of type *I* in [8].

In what follows, we will review type *III* of R-duals and present basic definitions of frames and Riesz basis; for more details, we refer the interested reader to the [2]. They are essential to our main result in the next section where by using the sequence  $\{S_\omega^{-1/2} \omega_j\}_{j \in I}$ , we will achieve the representation of positive square root of the inverse of the frame operator  $S_\omega$ .

Let  $\{f_i\}_{i \in I} \subset \mathcal{H}$  be a frame, with frame operator  $S_f$ . In [8], Stoeva and Christensen present various R-duals with focus on what they called R-dual of type *III* of  $\{f_i\}_{i \in I}$  as the sequence  $\{\omega_j\}_{j \in I}$  given by

$$\omega_j = \sum_{i \in I} \langle S_f^{-1/2} f_i, e_j \rangle Q h_i, \quad j \in I.$$

where  $Q$  is a bounded bijective operator on  $\mathcal{H}$  with  $\|Q\| \leq \sqrt{\|S_f\|}$  and  $\|Q^{-1}\| \leq \sqrt{\|S_f^{-1}\|}$ .

Among different R-duals presented in [8], type *III* R-dual is not symmetrical. If type *III* R-dual  $\{\omega_j\}_{j \in I}$  of a frame  $\{f_i\}_{i \in I}$  with respect to triple  $(\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, Q)$  is given, then it is not necessary that  $\{f_i\}_{i \in I}$  is an R-dual of type *III* of  $\{\omega_j\}_{j \in I}$ , (see [8]). In what follows,  $\mathcal{H}$  denote a separable Hilbert space, with inner product  $\langle \cdot, \cdot \rangle$ , and  $I$  be a countable index set.

**Definition 1.1.** [2] A collection of vectors  $\{f_i\}_{i \in I}$  in  $\mathcal{H}$  is a frame for  $\mathcal{H}$  if there exist constants  $A, B > 0$  such that

$$A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$

The constants  $A$  and  $B$  are called frame bounds. The frame is *A-tight*, if  $A = B$ . If at least the upper bounds  $B$  exists,  $\{f_i\}_{i \in I}$  is called a Bessel sequence. Here, the synthesis operator of  $\{f_i\}_{i \in I}$  is defined by  $T : \ell^2(I) \rightarrow \mathcal{H}, T\{c_i\}_{i \in I} = \sum_{i \in I} c_i f_i$ . Given a frame  $\{f_i\}_{i \in I}$  in  $\mathcal{H}$ , its frame operator is  $S_f = TT^* : \mathcal{H} \rightarrow \mathcal{H}, S_f f = \sum_{i \in I} \langle f, f_i \rangle f_i$ . In this case  $S_f$  is a bounded, invertible, self-adjoint and positive operator.

Moreover the sequence  $\{S_f^{-1}f_i\}_{i \in I}$  is also a frame for  $\mathcal{H}$  satisfying the reconstruction formula  $f = \sum_{i \in I} \langle f, S_f^{-1}f_i \rangle f_i$  for every  $f \in \mathcal{H}$ .

The sequence  $\{S_f^{-1}f_i\}_{i \in I}$  is called the canonical dual frame of  $\{f_i\}_{i \in I}$ . Also, any sequence  $\{g_i\}_{i \in I}$  in  $\mathcal{H}$  which is not the canonical dual and satisfies  $f = \sum_{i \in I} \langle f, f_i \rangle g_i = \sum_{i \in I} \langle f, g_i \rangle f_i$  is called an alternate dual frame of  $\{f_i\}_{i \in I}$ .

**Definition 1.2.** [2] A collection of vectors  $\{\omega_j\}_{j \in I}$  in  $\mathcal{H}$  is a Riesz sequence if there exist constants  $C, D > 0$  such that

$$C \sum_{j \in I} |c_j|^2 \leq \left\| \sum_{j \in I} c_j \omega_j \right\|^2 \leq D \sum_{j \in I} |c_j|^2$$

for all finite sequences  $\{c_j\}_{j \in I}$ . The numbers  $C$  and  $D$  are called Riesz bounds. A Riesz sequence  $\{\omega_j\}_{j \in I}$  is a Riesz basis for  $\mathcal{H}$  if  $\overline{\text{span}}\{\omega_j\}_{j \in I} = \mathcal{H}$ .

We now present the definition of the  $R$ -dual sequences of type  $I$  and  $III$ , stated in [1] and [8].

**Definition 1.3.** [1] Let  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$  be orthonormal bases for  $\mathcal{H}$ . Let  $\{f_i\}_{i \in I} \subset \mathcal{H}$  be such that

$$\sum_{i \in I} |\langle f_i, e_j \rangle|^2 < \infty, \quad \forall j \in I.$$

The  $R$ -dual of type  $I$  of  $\{f_i\}_{i \in I}$  with respect to the orthonormal bases  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$  is defined as the sequence given by

$$\omega_j = \sum_{i \in I} \langle f_i, e_j \rangle h_i, \quad j \in I.$$

**Definition 1.4.** [8] Let  $\{f_i\}_{i \in I}$  be a frame for  $\mathcal{H}$  with frame operator  $S_f$ . Let  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$  denote orthonormal bases for  $\mathcal{H}$  and  $Q : \mathcal{H} \rightarrow \mathcal{H}$  be a bounded bijective operator with  $\|Q\| \leq \sqrt{\|S_f\|}$  and  $\|Q^{-1}\| \leq \sqrt{\|S_f^{-1}\|}$ . The  $R$ -dual of type  $III$  of  $\{f_i\}_{i \in I}$  with respect to the triple  $(\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, Q)$ , is the sequence defined by

$$(1.1) \quad \omega_j = \sum_{i \in I} \langle S_f^{-1/2} f_i, e_j \rangle Q h_i, \quad j \in I.$$

The sequence  $\{f_i\}_{i \in I}$  in the above definition obtained in [8] as

$$(1.2) \quad f_i = \sum_{j \in I} \langle \omega_j, (Q^*)^{-1} h_i \rangle S_f^{1/2} e_j, \quad \forall i \in I.$$

**Proposition 1.5.** [8] *Let  $\{f_i\}_{i \in I}$  be a frame sequence and  $\{\omega_j\}_{j \in I}$  an R-dual of  $\{f_i\}_{i \in I}$  of type III. Then the following hold:*

- (i)  *$\{f_i\}_{i \in I}$  is a frame if and only if  $\{\omega_j\}_{j \in I}$  is a Riesz sequence; in the affirmative case the bounds for  $\{f_i\}_{i \in I}$  are also bounds for  $\{\omega_j\}_{j \in I}$ .*
- (ii)  *$\{f_i\}_{i \in I}$  is a Riesz sequence if and only if  $\{\omega_j\}_{j \in I}$  is a frame; in the affirmative case the bounds for  $\{f_i\}_{i \in I}$  are also bounds for  $\{\omega_j\}_{j \in I}$ .*
- (iii)  *$\{\omega_j\}_{j \in I}$  is a Riesz Basis if and only if  $\{f_i\}_{i \in I}$  is a Riesz Basis.*

Relation (1.2) does not imply, in general, that  $\{f_i\}_{i \in I}$  is an R-dual of type III of  $\{\omega_j\}_{j \in I}$ , that is, this definition of R-dual is not symmetric. With appropriate choice of  $Q$  the symmetry property of the sequences  $\{f_i\}_{i \in I}$  and  $\{\omega_j\}_{j \in I}$  will be preserved. The main purpose of this paper is to characterizing and representation of positive square root of the inverse of the frame operator  $S_\omega$  associated with  $\{\omega_j\}_{j \in I}$ . In order to analyze R-dual of type III in more moderate manner, we state the following basic lemma due to Stoeva and Christensen in [8].

**Lemma 1.6.** *Let  $V$  be a closed subspace of  $\mathcal{H}$  and  $\Phi : V \rightarrow V$  a bounded bijective operator. Define an extension of  $\Phi$  to an operator  $\tilde{\Phi} : V \rightarrow V$ ,  $\tilde{\Phi}(x_1 + x_2) = \Phi(x_1) + \|\Phi^{-1}\|^{-1} x_2$ ,  $x_1 \in V, x_2 \in V^\perp$*

*Then  $\tilde{\Phi}$  is bijective and bounded,  $\|\tilde{\Phi}\| = \|\Phi\|$ ,  $\|\tilde{\Phi}^{-1}\| = \|\Phi^{-1}\|$ , and  $\tilde{\Phi}^{-1}(x_1 + x_2) = \Phi^{-1}x_1 + \|\Phi^{-1}\| x_2$ ,  $x_1 \in V, x_2 \in V^\perp$ .*

*If  $\Phi$  is self-adjoint, then also  $\tilde{\Phi}$  is self-adjoint.*

## 2. MAIN RESULTS

In this section we continue with the setup in the previous section. Our first goal is to achieved the symmetrical R-duals of type III of a given frame  $\{f_i\}_{i \in I}$  and Riesz sequence  $\{\omega_j\}_{j \in I}$  in  $\mathcal{H}$ .

Let  $\{f_i\}_{i \in I}$  be a frame for  $\mathcal{H}$ , with frame operator  $S_f$  and synthesis operator  $T$ . Let  $\{\omega_j\}_{j \in I}$  be a Riesz sequence with frame operator  $S_\omega$  and the same optimal bounds as  $\{f_i\}_{i \in I}$ . Let

$$\dim(\ker T) = \dim(\text{span}\{\omega_j\}_{j \in I}^\perp),$$

since the synthesis operator for  $\{S_f^{-1/2}f_i\}_{i \in I}$  equals  $S_f^{-1/2}T$ , its kernel equals the kernel of  $T$ , and so

$$\dim(\ker(S_f^{-1/2}T)) = \dim(\ker T),$$

and

$$\dim(\text{span}\{\omega_j\}_{j \in I}^\perp) = \dim(\text{span}\{S_\omega^{-1/2}\omega_j\}_{j \in I}^\perp).$$

Then

$$\dim(\ker(S_f^{-1/2}T)) = \dim(\text{span}\{S_\omega^{-1/2}\omega_j\}_{j \in I}^\perp).$$

Now by Prop. 1.6 in [8], the sequence  $\{S_\omega^{-1/2}\omega_j\}_{j \in I}$  is R-dual of type I of  $\{S_f^{-1/2}f_i\}_{i \in I}$ , that is, there exist orthonormal bases  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$  so that

$$(2.1) \quad S_\omega^{-1/2}\omega_j = \sum_{i \in I} \langle S_f^{-1/2}f_i, e_j \rangle h_i, \quad j \in I.$$

Now consider the extension  $\widetilde{S_\omega^{1/2}}$  of  $S_\omega^{1/2}$  to an operator on  $\mathcal{H}$  as in Lemma (1.6), then  $\|\widetilde{S_\omega^{1/2}}\| = \|S_\omega^{1/2}\| \leq \sqrt{\|S_f\|}$  and  $\|(\widetilde{S_\omega^{1/2}})^{-1}\| \leq \sqrt{\|S_f^{-1}\|}$ . Apply the operator  $\widetilde{S_\omega^{1/2}}$  to (2.1) we get

$$(2.2) \quad \omega_j = \sum_{i \in I} \langle S_f^{-1/2}f_i, e_j \rangle \widetilde{S_\omega^{1/2}}h_i, \quad j \in I.$$

We now summarize what we have proved:

**Lemma 2.1.** *Let  $\{f_i\}_{i \in I}$  be a frame for  $\mathcal{H}$  and let  $\{\omega_j\}_{j \in I}$  be a Riesz sequence with the same optimal bounds as  $\{f_i\}_{i \in I}$ . Denote the synthesis operator for  $\{f_i\}_{i \in I}$  by  $T$  and the frame operators for  $\{f_i\}_{i \in I}$  and  $\{\omega_j\}_{j \in I}$  by  $S_f$  and  $S_\omega$ , respectively. If  $\dim(\ker T) = \dim(\text{span}\{\omega_j\}_{j \in I}^\perp)$  then there exists orthonormal bases  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$  for  $\mathcal{H}$  such that*

$$(2.3) \quad \omega_j = \sum_{i \in I} \langle S_f^{-1/2}f_i, e_j \rangle \widetilde{S_\omega^{1/2}}h_i, \quad j \in I,$$

where  $\widetilde{S_\omega^{1/2}}$  is a bounded bijective and self-adjoint extension of  $S_\omega^{1/2}$  on  $\mathcal{H}$ .

The sequence  $\{\omega_j\}_{j \in I}$  defined in (2.3) is called the symmetrical R-dual of type III of  $\{f_i\}_{i \in I}$  with respect to triple  $(\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, S_\omega^{1/2})$ .

The representation of  $\{\omega_j\}_{j \in I}$  in (2.3) is the main purpose our work, as it will be use for characterizing the operator  $S_\omega^{-1/2}$ .

Computation of the sequence  $\{f_i\}_{i \in I}$  in the following lemma preserve the symmetrical property of these sequences. Note that  $\{f_i\}_{i \in I}$  has many associated R-dual sequences for each choice of the orthonormal bases  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$ .

The following theorem is similar to Prop. 4.7 due to Diana and Christensen in [8]:

**Theorem 2.2.** *Let  $\{f_i\}_{i \in I}$  be a frame and  $\{\omega_j\}_{j \in I}$  an symmetrical R-dual of  $\{f_i\}_{i \in I}$  of type III, with respect to triple  $(\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, S_\omega^{1/2})$ . Denote the frame operator of  $\{f_i\}_{i \in I}$  and  $\{\omega_j\}_{j \in I}$  by  $S_f$  and  $S_\omega$ , respectively. Then the following hold:*

- (i)  $f_i = \sum_{j \in I} \langle \widetilde{S_\omega^{-1/2} \omega_j}, h_i \rangle S_f^{1/2} e_j, \quad \forall i \in I$ . i.e.  $\{f_i\}_{i \in I}$  is the symmetrical R-dual of type III of  $\{\omega_j\}_{j \in I}$  with respect to some triple  $(\{h_i\}_{i \in I}, \{e_i\}_{i \in I}, S_f^{1/2})$
- (ii) Let  $\{\gamma_j\}_{j \in I}$  denote the symmetrical R-dual of type III of the canonical dual frame  $\{S_f^{-1} f_i\}_{i \in I}$  with respect to some triple  $(\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, S_\omega^{1/2})$ . Then  $\{\omega_j\}_{j \in I}$  and  $\{\gamma_j\}_{j \in I}$  are biorthogonal.

*Proof.* (i) Using that  $\langle \widetilde{S_\omega^{-1/2} \omega_j}, h_i \rangle = \langle S_f^{-1/2} f_i, e_j \rangle$ . Now expanding  $S_f^{-1/2} f_i$  with respect to the orthonormal basis  $\{e_i\}_{i \in I}$  yields that  $S_f^{-1/2} f_i = \sum_{j \in I} \langle S_f^{-1/2} f_i, e_j \rangle e_j = \sum_{j \in I} \langle \widetilde{S_\omega^{-1/2} \omega_j}, h_i \rangle e_j$  and then

$$(2.4) \quad f_i = \sum_{j \in I} \langle \widetilde{S_\omega^{-1/2} \omega_j}, h_i \rangle S_f^{1/2} e_j, \quad \forall i \in I.$$

as claimed.

(ii) By definition  $\gamma_j = \sum_{i \in I} \langle S_f^{-1} f_i, S_f^{-1/2} e_j \rangle (S_\omega^{-1/2})$ , all calculation are similar to proof of theorem 4.7 in [8].  $\square$

Let  $\{f_i\}_{i \in I}$  be a frame and  $\{\omega_j\}_{j \in I}$  is a Riesz sequence in  $\mathcal{H}$ . In [5], Chuang and Zhao characterized the frame operator  $S_\omega$  and R-duals of type I of  $\{f_i\}_{i \in I}$  relating the synthesis operator  $T$  as follows:

**Theorem 2.3.** [5] *Let  $\{f_i\}_{i \in I}$  be a frame for  $\mathcal{H}$  and  $\{\omega_j\}_{j \in I}$  is a Riesz sequence in  $\mathcal{H}$ . Denote the synthesis operator for  $\{f_i\}_{i \in I}$  by  $T$ , the frame operator of  $\{f_i\}_{i \in I}$  by  $S_f$ , and the frame operator of  $\{\omega_j\}_{j \in I}$  by  $S_\omega$ . Then  $\{\omega_j\}_{j \in I}$  is an R-dual of type I of  $\{f_i\}_{i \in I}$  if and only if two condition hold:*

- (i) *there exists an antiunitary operator  $\Lambda, \mathcal{H} \rightarrow \overline{\text{span}}\{\omega_j\}_{j \in I}$  so that  $S_\omega = \Lambda S_f \Lambda^{-1}$ .*

$$(ii) \dim(\ker T) = \dim((\text{span}\{\omega_j\}_{j \in I})^\perp).$$

Let now  $\{\omega_j\}_{j \in I}$  be a symmetrical R-dual of type III of  $\{f_i\}_{i \in I}$  in  $\mathcal{H}$  and we define  $S_\omega^{-1/2}f = \sum_{k \in I} \langle f, S_\omega^{-1/2}\omega_k \rangle S_\omega^{-1/2}\omega_k$ . In the following theorem, let the countable index set  $I$  be the integer numbers set  $\mathbb{Z}$ , we obtain a representation for the operator  $S_\omega^{-1/2}$ :

**Theorem 2.4.** *Let  $\{f_i\}_{i \in I}$  be a frame for  $\mathcal{H}$  with frame operator  $S_f$  and  $\{\omega_j\}_{j \in I}$  be a symmetrical R-dual of type III of  $\{f_i\}_{i \in I}$  with triple  $(\{e_i\}_{i \in I}, \{h_i\}_{i \in I}, S_\omega^{1/2})$ , and frame operator  $S_\omega$  so that*

$$(2.5) \quad \sum_{j \in I} |\langle S_\omega^{-1/2}(h_0), h_j \rangle| < \infty$$

where  $\{e_i\}_{i \in I}$  and  $\{h_i\}_{i \in I}$  are orthonormal bases for  $\mathcal{H}$ . Then there exist operators  $\{\mathcal{V}_j\}_{j \in I}$  on  $\mathcal{H}$  such that

- (i)  $\mathcal{V}_j(S_\omega^{-1/2}h_0) = S_\omega^{-1/2}h_j$ ,
- (ii) If for all  $k \in I$ ,  $\{\mathcal{V}_j(h_k)\}_{j \in I}$  is a Bessel sequence with bound  $B$ , then there exist bounded operators  $\{\Lambda_k\}_{k \in \mathbb{Z}}$ , which satisfy  $\sup_{k \in \mathbb{Z}} \|\Lambda_k\| < \infty$  such that  $S_\omega^{-1/2}$  has the following representation

$$(2.6) \quad S_\omega^{-1/2}g = \sum_{k \in \mathbb{Z}} \langle S_f^{-1/2}f_k, S_f^{-1/2}f_0 \rangle \Lambda_k(g)$$

*Proof.* Let  $\mathcal{U} : \mathcal{H} \rightarrow \mathcal{H}$  be unitary shift operator  $\mathcal{U}h_i = h_{i+1}$  and  $\mathcal{V} : \mathcal{H} \rightarrow \mathcal{H}$  be the operator defined by  $\mathcal{V} = S_\omega^{-1/2}\mathcal{U}S_\omega^{1/2}$ . Then we defined  $\{\mathcal{V}_j\}_{j \in I}$  on  $\mathcal{H}$  by

$$\mathcal{V}_j := \mathcal{V}^j = S_\omega^{-1/2}\mathcal{U}^j S_\omega^{1/2}$$

Then (i) is obviously fulfilled, since

$$\mathcal{V}_j(S_\omega^{-1/2}h_0) = \mathcal{V}^j(S_\omega^{-1/2}h_0) = S_\omega^{-1/2}\mathcal{U}^j h_0 = S_\omega^{-1/2}h_j, \quad \forall j \in I.$$

For the proof of (ii)

$$\begin{aligned} S_\omega^{-1/2}g &= \sum_{j \in I} \langle g, h_j \rangle S_\omega^{-1/2}(h_j) \\ &= \sum_{j \in I} \langle g, h_j \rangle \mathcal{V}_j S_\omega^{-1/2}(h_0) \\ &= \sum_{j \in I} \langle g, h_j \rangle \mathcal{V}_j \left( \sum_{i \in I} \langle S_\omega^{-1/2}(h_0), h_i \rangle h_i \right) \\ &= \sum_{i \in I} \langle S_\omega^{-1/2}(h_0), h_i \rangle \sum_{j \in I} \langle g, h_j \rangle \mathcal{V}_j(h_i) \end{aligned}$$

Now  $\langle S_\omega^{-1/2}(h_0), h_i \rangle = \sum_{k \in I} \langle h_0, S_\omega^{-1/2} \omega_k \rangle \langle S_\omega^{-1/2} \omega_k, h_i \rangle$   
and then

$$\begin{aligned} S_\omega^{-1/2} g &= \sum_{i \in I} \left[ \sum_{k \in I} \langle h_0, S_\omega^{-1/2} \omega_k \rangle \langle S_\omega^{-1/2} \omega_k, h_i \rangle \right] \sum_{j \in I} \langle g, h_j \rangle \mathcal{V}_j(h_i) \\ &= \sum_{i \in I} \langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle \sum_{j \in I} \langle g, h_j \rangle \mathcal{V}_j(h_i) \end{aligned}$$

Define  $\Lambda_k$  by  $\Lambda_k(g) = \sum_{j \in I} \langle g, h_j \rangle \mathcal{V}_j(h_i)$ . Then

$$S_\omega^{-1/2} g = \sum_{i \in I} \langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle \Lambda_k(g)$$

For convergence, the sequence  $\{\mathcal{V}_j(h_i)\}_{j \in I}$  is a Bessel sequence with bound  $B$  and so the series defining  $\Lambda_k$  are unconditionally convergent and  $\|\Lambda_k\| \leq \sqrt{B}$ ,  $\forall k \in \mathbb{Z}$ :

$$\left\| \sum_{j \in I} \langle g, h_j \rangle \mathcal{V}_j(h_i) \right\|^2 \leq B \sum_{j \in I} |\langle g, h_j \rangle|^2 = B \|g\|^2.$$

On the other hand, the series in the construction of  $\Lambda_k$  are unconditionally convergent and then the operators converge to  $S_\omega^{-1/2}$  unconditionally in the operator norm, that is for finite subsets  $I_1$  and  $I_2$  of  $I$ , from our above calculations,

$$\begin{aligned} \left\| \sum_{i \in I_1} \langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle \sum_{j \in I_2} \langle g, h_j \rangle \mathcal{V}_j(h_i) \right\| &= \\ \left\| \sum_{i \in I_1} \langle S_\omega^{-1/2}(h_0), h_i \rangle \sum_{j \in I_2} \langle g, h_j \rangle \mathcal{V}_j(h_i) \right\| &\leq \\ \left\{ \sum_{i \in I_1} |\langle S_\omega^{-1/2}(h_0), h_i \rangle| \right\} \left\| \sum_{j \in I_2} \langle g, h_j \rangle \mathcal{V}_j(h_i) \right\| &\leq \\ \left\{ \sum_{i \in I_1} |\langle S_\omega^{-1/2}(h_0), h_i \rangle| \right\} \sqrt{B} \|g\|. & \end{aligned}$$



By (2.5), the series in the construction of  $\Lambda_k$  are unconditionally convergent. Finally, for a finite  $J \subset I$ , we have:

$$\begin{aligned}
& \|S_\omega^{-1/2} - \sum_{i \in J} \langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle \Lambda_i\| = \\
& \sup_{\|g\|=1} \|S_\omega^{-1/2}(g) - \sum_{i \in J} \langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle \Lambda_i(g)\| = \\
& \sup_{\|g\|=1} \left\| \sum_{i \in J^c} \langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle \Lambda_i(g) \right\| \leq \\
& \left\{ \sum_{i \in J^c} |\langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle| \right\} \sup_{\|g\|=1} \sup_{i \in J^c} \|\Lambda_i(g)\| \leq \\
& \left\{ \sum_{i \in J^c} |\langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle| \right\} \sqrt{B}
\end{aligned}$$

Now using (2.5) and that  $|\langle S_\omega^{-1/2} h_0, h_i \rangle| = |\langle S_f^{-1/2} f_i, S_f^{-1/2} f_0 \rangle|$ , it follows that the operators converge to  $S_\omega^{-1/2}$  unconditionally in the operator norm.  $\square$

#### REFERENCES

- [1] P. Casazza, G. Kutyniok, and M. Lammers, *Duality principles in frame theory*, J. Fourier Anal. Appl. **10**, 383-408, (2004).
- [2] O. Christensen, *Frame and Bases, An Introductory Course*, Birkhäuser, Basel (2008).
- [3] O. Christensen, H.O. Kim, R.Y. Kim, *On the duality principle by Casazza, Kutyniok, and lammers*, J. Fourier Anal. Appl. **17**, 640-655, (2011).
- [4] O. Christensen, X.C. Xiao, Y.C. Zhu, *Characterizing R-duality in Banach spaces*, Acta Math. Sin. Engl. Ser. **1**, 75-84, (2013).
- [5] Z. Chuang, J. Zhao, *On equivalent conditions of two sequences to be R-dual*, J. Inequal. Appl., 2015:10, 1-8, (2015).
- [6] I. Daubechies, H.J. Landau, Z. Landau, *Gabor time-frequency lattices and the Waxler-Raz identity*, J. Fourier Anal. Appl. **1**, 437-478, (1995).
- [7] K. Gröchenig, *Foundations of time-Frequency Analysis*, Birkhäuser, Boston (2000).
- [8] D.T. Stoeva, O. Christensen, *On R-duals and the Duality principle in Gabor Analysis*, J. Fourier Anal. Appl. **21**, 383-400, (2015).

DEPARTMENT OF MATHEMATICS, PAYAME NOOR UNIVERSITY, P.O.Box 19395-3697, TEHRAN, IRAN

*E-mail address:* j\_cheshmavar@pnu.ac.ir

*E-mail address:* aliakbarnia52@yahoo.com